MATH 1650: SECTION 3.1: INTRODUCTION TO RATIONAL FUNCTIONS

RATIONAL FUNCTION: A **rational function** r is a **ratio** of polynomial functions. That is, r has the form:

$$r(x) = \frac{p(x)}{q(x)},$$

where p and q are polynomial functions.

NOTES:

- Dividing by $q(x)$ could cause issues with the domain. (why?)

STRATEGY: To find the domain of a rational function, we first set the **denominator** equal to 0 and solve to find **excluded** values. The domain consists of all real numbers **except** the excluded values.

EXAMPLE: Let $f(x) = \dfrac{2x^2 - 3x - 2}{x^2 - 4}$.

- Find the values excluded from the domain.

  We set the denominator $x^2 - 4 = 0$ and solve. We get $x^2 = 4$ or $x = \pm\sqrt{4} = \pm 2$.

- Write the domain using interval notation.

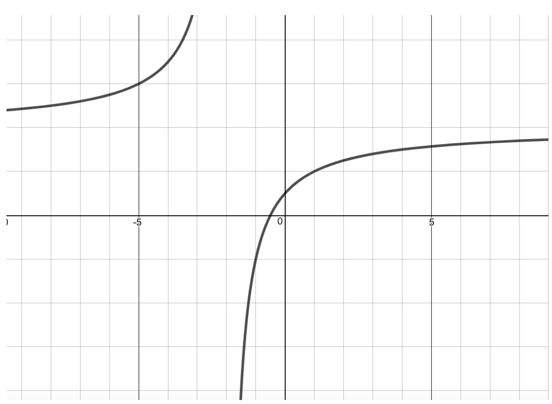
  The domain is $\{x \mid x \neq \pm 2\}$ which, in interval notation, is: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

- Use a graphing utility to examine the behavior of the the graph of the function near the excluded values.

  Graphing f using desmos produces the graph below on the left. The graph appears to be 'torn' at $x = -2$ creating what is known as a **vertical asymptote**.

  Using a table of values, we see that as we evaluate $f(x)$ at values of x a bit less than -2, that is, numbers just to the **left** of -2, the function values appear to increase without bound. In this case we write 'as $x \to -2^-$, $f(x) \to \infty$' and say 'as x approaches -2 **from the left**, $f(x)$ approaches infinity.'

  On the other hand, as we evaluate f at values of x a bit greater than $x = -2$, that is, numbers just to the the **right** of -2, we obtain values that grow more and more negative. Here, we write 'as $x \to -2^+$, $f(x) \to -\infty$' and say 'as x approaches -2 **from the right**, $f(x)$ approaches negative infinity.'



x_1	$f(x_1)$
-2.1	32
-2.01	302
-2.001	3002
-2.0001	30002
-2.00001	300002
-1.9	-28
-1.99	-298
-1.999	-2998
-1.9999	-29998

One way to remember what the notation '$x \to -2^-$' means is to think '$x = -2 -$ a little bit.' Numbers like $x = -2 - 0.1 = -2.1$, $x = -2 - 0.01 = -2.01$, etc.

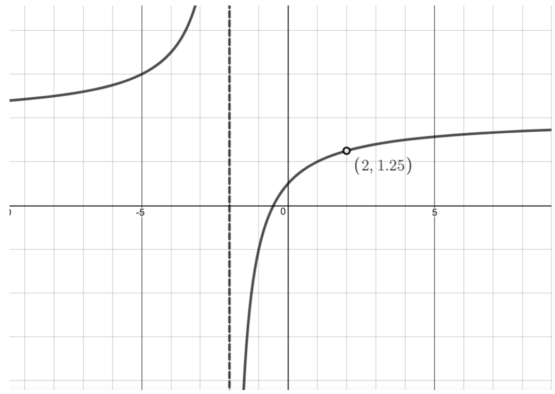
Likewise, we think of $x \to -2^+$ as indicating numbers '$x = -2 +$ a little bit' so we are considering numbers like $x = -2 + 0.1 = -1.9$, $x = -2 + 0.01 = -1.99$, etc.

Near $x = 2$ the graph of $y = f(x)$ seems (eerily) calm. Once again, we make a table of values and it appears as if as x gets close to 2 from either direction, $f(x)$ appears to gets closer and closer to 1.25. We write 'as $x \to 2$, $f(x) \to 1.25$' and say 'as x approaches 2, $f(x)$ approaches 1.25.' Since f is not defined when $x = 2$, we draw a 'hole' on the graph at $(2, 1.25)$.

Given our analysis, we refine the graph of $y = f(x)$ as seen below on the right. Note we indicate the vertical asymptote using a dashed line and use an 'open circle' to indicate the hole in the graph.

$$f(x) = \frac{2x^2 - 3x - 2}{x^2 - 4}$$

x_1	$f(x_1)$
1.9	1.2307692
1.99	1.2481203
1.999	1.2498125
1.9999	1.2499812
2.1	1.2682927
2.01	1.2518703
2.001	1.2501875
2.0001	1.2500187



EXAMPLE: For each of the functions below:

- Find the values excluded from the domain.
- Write the domain using interval notation.
- Use a graphing utility to examine the behavior of the the graph of each function near the excluded values.

$$f(x) = \frac{3x - 1}{x^2 - 25}$$

- excluded values:
- domain:
- Behavior near excluded values:

$$g(x) = \frac{2x - 8}{x^2 - x - 12}$$

- excluded values:
- domain:
- Behavior near excluded values:

Determining the y-coordinate of the hole in the graph in the last example could be problematic if we are using tables. If we analyze the function a bit more, we realize:

$$g(x) = \frac{2x-8}{x^2-x-12} = \frac{2(x-4)}{(x-4)(x+3)} = \frac{2\cancel{(x-4)}}{\cancel{(x-4)}(x+3)} = \frac{2}{x+3},$$

where the last equality holds provided $x \neq 4$ (otherwise, we'd be dividing by 0.)

Hence if $x \approx 4$, then $g(x) \approx \dfrac{2}{4+3} = \dfrac{2}{7} = 0.\overline{285714}$. This means the hole in the graph is located at $\left(4, \dfrac{2}{7}\right)$.

EXAMPLE: Let $f(x) = \dfrac{x^2-4}{x^2+x-2}$.

- Find the values excluded from the domain of f.

- Write the domain of f using interval notation.

- Use a graphing utility to help you describe the behavior of f near excluded values.

- Reduce $f(x)$ to lowest terms in order to exactly determine the location of the hole in the graph of $y = f(x)$.

IN GENERAL: At **excluded values**, the graph of a rational function has either a **vertical asymptote** or a **hole**.
Holes in the graph come from **factors cancelling** from the denominator.
If the graph has a **hole**, **reduce** the function to **lowest terms** to help find the y-coordinate of the hole.

EXAMPLE: Determine the location of vertical asymptotes or holes in the graph of each of the following functions:

- $f(x) = \dfrac{x^2 + x - 12}{2x^2 - 7x - 4}$

- $g(x) = \dfrac{x^2 - x - 12}{2x^2 - 7x - 4}$

END BEHAVIOR: For polynomial functions, end behavior is determined by the leading term. Since rational functions are ratios of polynomials, we use the leading term test as applied both to the numerator and the denominator to help us analyze the end behavior of rational functions,

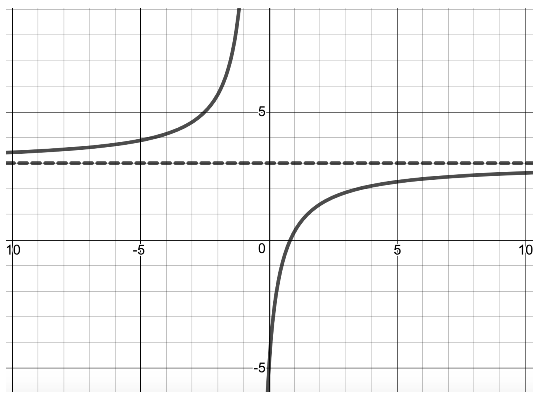
EXAMPLE: Analyze the end behavior of the following functions algebraically.
Confirm your suspicions using a graphing utility.

- $f(x) = \dfrac{6x-5}{2x+1}$

  To analyze what happens to $f(x) = \dfrac{6x-5}{2x+1}$ as $x \to \pm\infty$, we apply the leading term test to both the numerator and denominator: as $x \to \pm\infty$, $f(x) = \dfrac{6x-5}{2x+1} \approx \dfrac{6x}{2x} = 3$. Hence, as $x \to \pm\infty$, $f(x) \approx 3$.

  This means the horizontal line $y = 3$ is a **horizontal asymptote** to the graph of $y = f(x)$.

  We confirm this graphically using desmos below.



- $r(x) = \dfrac{3x-1}{x^2-25}$

  Once again our strategy is to apply the leading term test to the numerator and denominator of $r(x)$.

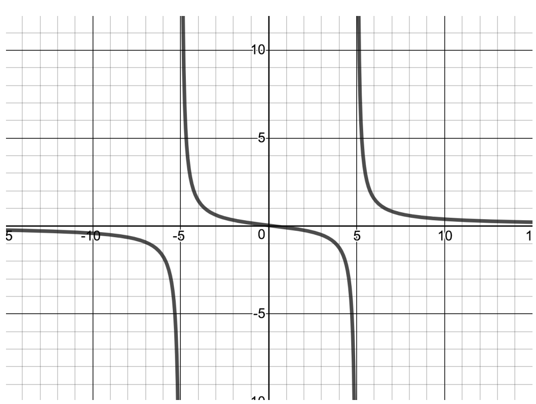
$$r(x) = \frac{3x-1}{x^2-25} \approx \frac{3x}{x^2} = \frac{3}{x}$$

  So now, we need to analyze what happens to $\dfrac{3}{x}$ as $x \to \pm\infty$. We imagine substituting numbers such as $x = \pm 100$, $x = \pm 1000$, $x = \pm 10000$, etc. and get $\dfrac{3}{\pm 100}, \dfrac{3}{\pm 1000}, \dfrac{3}{\pm 10000}$, etc.

  Hence the values of $\dfrac{3}{x}$ are steadily approaching 0.

  Geometrically, we see the graph of $y = r(x)$ getting closer and closer to the x-axis ($y = 0$) as $x \to \pm\infty$.

  As such, we say $y = 0$ is a **horizontal asymptote** to the graph of $y = r(x)$.



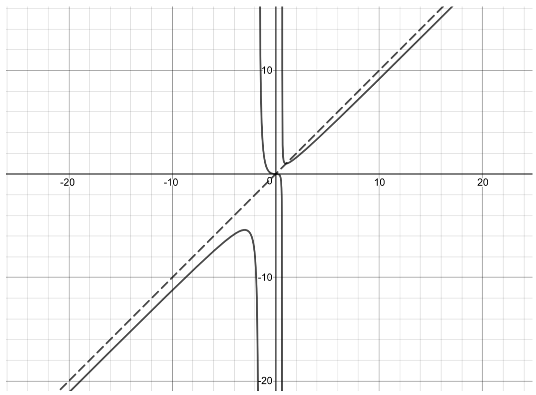
EXAMPLE: Analyze the end behavior of $h(x) = \dfrac{x^3}{x^2+x-1}$ algebraically.
Confirm your suspicions using a graphing utility.

Applying the leading term test to the numerator and denominator of $h(x)$, we find that as $x \to \pm\infty$,

$$h(x) = \frac{x^3}{x^2+x-1} \approx \frac{x^3}{x^2} = x.$$

This suggests that the graph of $y = h(x)$ should approach the graph of $y = x$ as $x \to \pm\infty$.

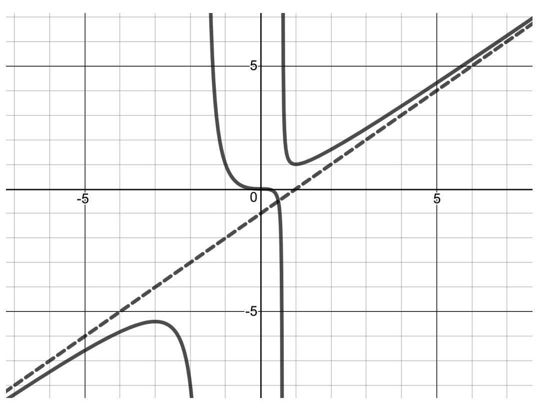
However, graphing both $y = h(x)$ and $y = x$ on desmos reveals the graphs do **not** grow closer together as $x \to \pm\infty$, but rather, become **parallel**.



Doing some more algebra (long division), we get $h(x) = \dfrac{x^3}{x^2+x-1} = x - 1 + \dfrac{2x-1}{x^2+x-1}$.

As $x \to \pm\infty$, $\dfrac{2x-1}{x^2+x-1} \approx \dfrac{2x}{x^2} = \dfrac{2}{x} \to 0$. Hence, $h(x) = \dfrac{x^3}{x^2+x-1} = x - 1 + \dfrac{2x-1}{x^2+x-1} \approx x - 1$.

Graphing $y = h(x)$ along with $y = x - 1$ shows these two graphs grow indistinguishable as $x \to \pm\infty$, indicating $y = x - 1$ is a **slant asymptote** to the graph of $y = h(x)$.



END BEHAVIOR OF RATIONAL FUNCTIONS: As a general rule, use the leading term of the numerator and the leading term of the denominator to help you determine the end behavior of a rational function.

More specifically:

- **HORIZONTAL ASYMPTOTES** occur in one of two cases:
  - If the degree of the numerator is **less** than the degree of the denominator, the HA is $y = 0$.
  - If the degree of the numerator is the **same** as the degree of the denominator, the HA is:

$$y = \frac{\text{leading coefficient of the numerator}}{\text{leading coefficient of the denominator}}$$

- **SLANT (OBLIQUE) ASYMPTOTES** occur if the degree of the numerator is **exactly one more** than the degree of the denominator. To find the asymptote, you need to perform **long division**.

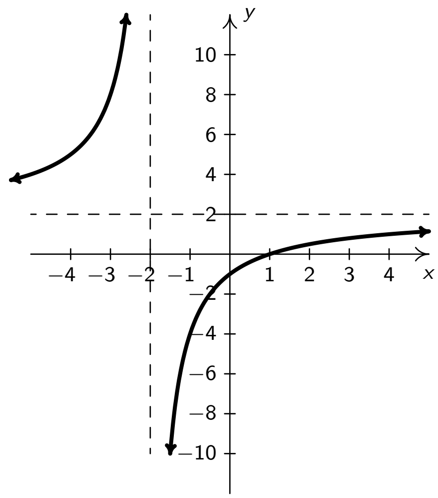
EXAMPLE: Find the horizontal or slant asymptote of the following rational functions algebraically.
Check your answer using a graphing utility.

- $f(x) = \dfrac{3 - x}{2x + 1}$

- $g(x) = \dfrac{5x}{x^2 + 1}$

$$r(x) = \frac{2x^3 - x^2 + 2x}{x^2 + 1}$$

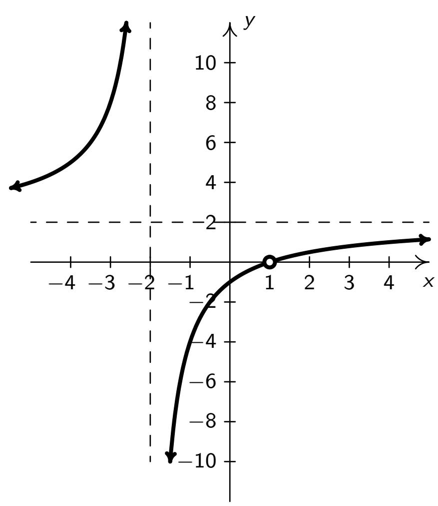
EXAMPLE: Find numbers a, b, and c so that the graph of $f(x) = a\dfrac{x-b}{x+c}$ matches the graph below.



$y = f(x)$

EXAMPLE: Find a formula for the function $g(x)$ whose graph matches the graph below.

HINT: Start with your answer for $f(x)$ from the previous problem and modify it.



$y = g(x)$

HOMEWORK: Section 3.1: 15 - 37 odd